MODULATIONAL INSTABILITY AND EXTREME WAVES IN WATER OF FINITE DEPTH

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Experiments in the Ocean Wave Basing at MARINTEK were carried out to study the instability of a plane wave to oblique side band perturbations in finite water depth. Observations, with the support of numerical simulations, confirm that a carrier wave becomes modulationally unstable even for relative water depths \( k_0h < 1.36 \) (with \( k \) the wavenumber of the plane wave and \( h \) the water depth), if it is perturbed by appropriate oblique disturbances. Results suggest that the underlying mechanism is still a plausible explanation for the generation of freak waves in water of finite depth.

1. INTRODUCTION

The occurrence of extreme waves plays a substantial role in many branches of physics and engineering. During the past two decades, there has been a rapid increase in the understanding of the mechanisms leading to the formation of extreme waves in the open ocean. Apart from the linear wave superposition and the effects of currents on waves (caustic theory), the instability of deep-water wave trains to side band perturbations has been found to be a relevant mechanism for the formation of extreme waves in long-crested sea state conditions (i.e. unidirectional wave fields). This mechanism, which is basically a generalization of the Benjamin–Feir instability or modulational instability (Zakharov & Ostrovsky, 2009), can be conveniently described by deep-water third-order nonlinearity. The main finding is that modulational instability increases the probability of occurrence of extreme wave heights (defined as a wave height larger than twice the significant wave height) by almost one order of magnitude if compared with linear theory (Onorato et al., 2006).

The previous results have been obtained for deep-water waves only. In conditions of arbitrary water depth, on the other hand, finite-amplitude waves generate a wave-induced current and hence less and less energy is made available for nonlinear processes. As a result, modulational instability gradually vanishes with decreasing relative water depth \( kh \) (where \( k \) is the wave-number and \( h \) is the water depth) and eventually ceases to exist for \( kh < 1.36 \). This well known result was obtained theoretically by Benjamin (Benjamin, 1967) and Whitham (Whitham, 1974) for long crested-wave fields. Note that, on the contrary to what is observed in deep water, not only may the suppression of modulational instability in finite water depth have implication for wave statistics (i.e. deviations from Gaussianity are substantially reduced), but it may also play a substantial role for spectral wave modelling as there is a considerable reduction of the nonlinear energy transfer rates (Janssen & Onorato, 2007). Under these circumstances, therefore, the energy balance would be determined by wind input and dissipation predominantly, while the contribution of nonlinear interactions could be neglected. This outcome, however, is valid only if third-order nonlinearity is considered. It has been suspected, nonetheless, that nonlinearities higher than the third-order may actually trigger wave instability and hence sustain
nonlinear interactions also when \( kh < 1.36 \) (Kristiansen et al., 2005). Furthermore, the limit of \( kh < 1.36 \) is valid for long crested waves only. In directional wave fields, unstable disturbances may also propagate obliquely to the direction of the carrier wave. In this respect, when the depth decreases, the instability area becomes narrower and its increment decreases and eventually vanishes for collinear disturbances, but still persists for oblique perturbations (Kharif and Pelinovsky, 2003). Thus, the directional width may in principle sustain nonlinear modulational instability in shallow and intermediate (finite) water depth (Francius & Kharif, 2006).

In the present conference proceeding, we discuss a set of laboratory experiments that were carried out in the Ocean Wave Basin at MARINTEK (Norway) to investigate wave dynamics in finite water depth. The aim is to validate the conjecture that the modulation of plane wave by oblique perturbations can lead to rogue waves also when \( kh < 1.36 \). Detailed post processing of the data and results, including a comparison with numerical simulations, can be found in Toffoli et al. (2013).

2. EXPERIMENTAL SETUP AND RESULTS

Experiments were conducted in the directional ocean wave basin at MARINTEK (Norway), which is 70 m wide and 50 m long. The facility is equipped with 144 individually controlled flaps for the generation of directional wave components on the 70 m side and a unidirectional wavemaker on the 50 m one (only the former was used for the present study); absorbing beaches are mounted on the opposite side of each wavemaker. The water depth is uniform and controlled by a movable bottom, which was set to a depth \( h = 0.78 \) m for this specific experiment. The surface elevation was monitored by 25 capacitance gauges distributed along the basin at a sampling frequency of 200 Hz; three 3-probe arrays shaped as a triangle and one 6-probe array shaped as a pentagon with a probe in the middle were also deployed to monitor directional properties (a schematic diagram of the experiment is presented in Figure 1).

Initial conditions consisted in a sinusoidal (carrier) wave, which was seeded by 4 oblique side bands. Different configurations of the carrier wave were selected. Here we report \( T = 1.35 \) s and 1.68 s, which defined relative water depths \( k_0 h = 1.78, 1.24 \), respectively. The related wave amplitudes were selected such that all tests ran with identical wave steepness \( k_{0\text{lab}} = 0.14 \). Perturbations were accurately chosen within the unstable region of the instability diagram. Specifically, two upper and two lower oblique disturbances with coordinates \( [\Delta k_x, \Delta k_y], [\Delta k_x, -\Delta k_y], [-\Delta k_x, -\Delta k_y], \) and \( [-\Delta k_x, \Delta k_y] \) were applied (see Figure 1b).

The modulational wavenumber components along the mean wave direction (x) were selected by ensuring a number of 5 waves under the modulation in the physical space and thus about 10 waves in the time domain. Components in the transverse direction (y) were chosen such that \( |\Delta k_x/\Delta k_y| = 0.6 \) for \( k_0 h > 1.36 \) and \( |\Delta k_x/\Delta k_y| = 0.75 \) for \( k_0 h < 1.36 \). Tests were also repeated for collinear configurations (i.e., unidirectional propagation) by imposing \( \Delta k_y = 0 \). Each perturbation was then given a rather large amplitude equivalent to 30% of the amplitude of the carrier wave. This is related to the difficulties in replicating wave instability in finite water depth due to the large space scales that are required to fully develop the maximum wave amplification and the relatively short length of the facility. The selection of such large amplitudes coincides with an advanced stage of the modulation process, which is expected to have started under the effect of infinitesimal (small-amplitude) disturbances.

For \( k_0 h = 1.78 \), we observed a rapid asymmetric growth of the side bands both in one and two-dimensional propagation. There is, nonetheless, an energy loss for higher frequency components due to selective breaking towards the end of the basin, which is consistent with previous observations in, e.g., (Tulin & Waseda, 1999). On the contrary, for \( k_0 h < 1.36 \), energy transfer ceases under the condition of unidirectional propagation (see Figure 2), at least within the boundaries of the facility, in agreement with the suppression of modulational instability (Benjamin, 1967). A certain degree of dissipation was, however, recorded along the basin as a result of bottom friction and depth-induced breaking. By allowing propagation in two horizontal dimensions, on the other hand, the imposed oblique perturbations trigger modulational instability and consequently a nonlinear energy transfer between the carrier and the perturbations.
In the physical space, wave instability results in a nonlinear energy focussing, which makes the modulation more pronounced. This is highlighted in Figure 3, where the evolution of a carrier wave perturbed by oblique side bands (for both $k_0 h = 1.78$ and 1.24) is presented. As modulation instability only relates to free wave components, bound waves were filtered out by removing frequencies lower than 0.5 and larger than 1.5 times the peak frequency. The resulting time series indicate clearly the enhancement of wave modulation and the consequent amplitude growth. This is more pronounced for $k_0 h = 1.78$, where waves become particularly steep and eventually start breaking towards the end of the basin.

Figure 1. Schematic representation of the experimental set up (a) and initial condition imposed at the wavemaker (b).
Figure 2. Spectral evolution along the basin for relative water depth $k_0h = 1.24$: test with collinear perturbations (upper panels); test with oblique perturbations (lower panels).

Figure 3. Spatial evolution of a wave packed perturbed by an oblique modulation: (a) $k_0h=1.78$; (b) $k_0h=1.24$. 
3. COMPARISON WITH NUMERICAL SIMULATIONS

In order to provide a theoretical benchmark for the interpretation of the experimental results, we replicated laboratory observations with direct numerical simulations of the Euler equations, by applying the Higher Order Spectral Method (HOSM) proposed by (West et al., 1987). Details for the application of HOSM in finite water depth can be found, for example, in (Toffoli et al., 2009). The method uses a series expansion in the wave slope of the vertical velocity to resolve the surface elevation and the velocity potential. Simulations have been performed with a third order expansion so that only four-wave interaction effects were enabled (Tanaka, 2001). No dissipation term was applied. One of the drawbacks of the numerical method is that it replicates the temporal evolution of an initial surface, while experiments describe the spatial evolution of an initial time series. To allow a comparison, we make the assumption that time can be translated into space by means of the group velocity. This approach has been successfully used to compare experimental data with HOSM simulations in (Toffoli et al., 2010).

For $k_0 h > 1.36$, numerical simulations show that modulational instability induces a rapid and substantial growth of wave amplitude in both one and two dimensional propagation. It is worth mentioning that wave growth is slightly quicker in the collinear (unidirectional) configuration, though. Experimental data are also in good qualitative agreement with model predictions (see Figure 4 for a direct comparison). The latter, nonetheless, seems to slightly underestimate the wave amplitude in the defocussing stage for collinear modulations.

For $k_0 h < 1.36$, the modulation is suppressed for collinear perturbations and wave amplitude does not change significantly throughout the basin. This is predicted numerically and confirmed experimentally, see Figure 5 for this particular case. It is worth noting that the experimental set up assumes, a priori, an advanced stage of modulation as initial condition. Hence, during the first wavelengths of propagation, a defocussing takes place, reducing the extent of the modulation and wave amplitude. In the presence of oblique perturbations, on the other hand, wave packets remain unstable. Numerical simulations, in this respect, confirm that this instability still leads to the development of robust wave amplification. Remarkably, this numerical result is qualitatively consistent with experimental observations.

![Figure 4](image.png)

Figure 4. Spatial evolution of the maximum wave amplitude for relative water depth $k_0 h=1.78$. 

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### CONCLUSIONS

In conclusion, we have discussed a set of laboratory experiments in a large directional wave basin to investigate the instability of a sinusoidal (carrier) wave to oblique perturbations in finite water depth. Although modulational instability is suppressed for \( k_0h < 1.36 \) when propagation is restricted to one dimension, experiments show that perturbations propagating at an angle with the carrier waves can still trigger wave modulation, resulting in an amplitude growth (up to twice the amplitude of the carrier wave) also when \( k_0h < 1.36 \). Experimental evidence was supported and confirmed by direct numerical simulations of the Euler equations, which describes the evolution of the experimental wave packets under the effect of a four-wave interaction process only.

### ACKNOWLEDGEMENT

This work has been supported by European Community's Seventh Framework Programme through the grant to the budget of the Integrating Activity HYDRALAB IV within the Transnational Access Activities, Contract no. 261520. F.W.O. Project G.0333.09 and E.U. Project EXTREME SEAS (Contract No. SCP8-GA-2009-234175) are also acknowledged. L.F. and J.M. acknowledge the Hercules Foundation and the Flemish Government department EWI for providing access to the Flemish Supercomputer Center. L.C. acknowledges the support of the MyWave EU funded FP7-SPACE-2011-1/CP-FP Project.

### REFERENCES


